

A VARIATIONAL PROBLEM FOR THE SPATIAL SEGREGATION OF REACTION–DIFFUSION SYSTEMS ^{*}

Monica Conti [†], Susanna Terracini[‡] and Gianmaria Verzini [§]

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Abstract

In this paper we study a class of stationary states for reaction–diffusion systems of $k \geq 3$ densities having disjoint supports. For a class of segregation states governed by a variational principle we prove existence and provide conditions for uniqueness. Some qualitative properties and the local regularity both of the densities and of their free boundaries are established in the more general context of a functional class characterized by differential inequalities.

1 Introduction

The occurrence of nontrivial steady states (pattern formation) for reaction–diffusion systems has been widely studied in the literature. Of particular interest is the existence of spatially inhomogeneous solutions for competition models of Lotka–Volterra type. This study has been carried out mainly in the case of two competing species, see e.g. [4, 10, 19, 20, 22, 23, 26, 27, 28]; in recent years also the case of three competing densities, which is far more complex, has become object of an extensive investigation [13, 14, 24, 25]. In most cases, the pattern formation is driven by the presence of different diffusion rates when the coefficients of intra–specific and inter–specific competitions are suitable related. A remarkable limit case of pattern formation yields to the segregation of competing species, that is, configurations where different densities have disjoint habitats; see [11, 12, 16, 17, 18, 29]. Object of the present paper is to study a class of possible segregation states, involving an arbitrary number of competing densities, which are governed by a minimization principle rather than competition–diffusion. Roughly speaking, we are going to deal with stationary

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[†]Dipartimento di Matematica del Politecnico di Milano, Piazza Leonardo da Vinci, 32; 20133 Milano, Italy. email: *monica@mate.polimi.it*

[‡]Dipartimento di Matematica e Applicazioni Universit degli Studi di Milano–Bicocca Via Bicocca degli Arcimboldi, 8; 20126 Milano, Italy. email: *suster@matapp.unimib.it*

[§]Dipartimento di Matematica del Politecnico di Milano, Piazza Leonardo da Vinci, 32; 20133 Milano, Italy. email: *giaver@mate.polimi.it*

configurations of $k \geq 2$ densities that interact only through the boundaries of their nodal sets; the minimization involves the sum of the internal energies, with the constraint of being segregated states. In other words, the supports of the densities have to satisfy a suitable optimal partition problem in \mathbb{R}^N . Precisely, let Ω be a bounded open subset of \mathbb{R}^N ($N \geq 2$) and let us call *segregated state* a k -uple $U = (u_1, \dots, u_k) \in (H^1(\Omega))^k$ where

$$u_i(x) \cdot u_j(x) = 0 \quad i \neq j, \text{ a.e. } x \in \Omega.$$

We define the internal energy of U as

$$J(U) = \sum_{i=1, \dots, k} \left\{ \int_{\Omega} \left(\frac{1}{2} d_i^2(x) |\nabla u_i(x)|^2 - F_i(x, u_i(x)) \right) dx \right\},$$

where the F_i 's satisfy suitable assumptions (see (A1), (A2) below). Our first goal is to minimize J among a class of segregated states subject to some boundary and positivity conditions; next we shall develop a regularity and a free boundary theory for minimizers. In performing the second goal the main tools will come from recent results that Caffarelli, Jerison and Koenig [6] (see also [2, 3, 5, 7] and references therein) have obtained in the study of free boundaries in other contexts. Recently, free boundary problems have been studied in connection with the asymptotic behaviour of some models of population dynamics with diffusion as in Dancer and Du ([15]) and Dancer, Hilhorst et al in ([17]), in the case of two competing species. In a forthcoming paper we shall show how our variational problem appears as a limiting problem for some classes of competition–diffusion problems.

Our first result establishes existence for this problem; then we discuss the uniqueness of the solution. Surprisingly enough, the minimizer can be proven to be unique for a large class of Lagrangians. A remarkable fact is that solutions to this variational problem satisfy extremality conditions in the form of differential inequalities of special type. Precisely they belong to the functional class

$$\mathcal{S} = \left\{ (u_1, \dots, u_h) \in (H^1(\Omega))^k : \begin{array}{l} u_i \geq 0, u_i \cdot u_j = 0 \text{ if } i \neq j \\ -\Delta u_i \leq f_i(x, u_i), -\Delta \hat{u}_i \geq \hat{f}(x, \hat{u}_i) \end{array} \right\}$$

where $\hat{u}_i = u_i - \sum_{h \neq i} u_h$ and $\hat{f}(x, \hat{u}_i) = \sum_j f_j(x, \hat{u}_i) \chi_{\{u_j > 0\}}$.

A further reason of interest in the class \mathcal{S} is that it contains also the asymptotic limits of the solutions of a large class of competition–diffusion systems when the inter-specific competition terms tend to infinity. This will be the object of a forthcoming paper; a link between some variational problems and competing species systems has been traced by the authors in [8, 9].

An important part of the paper is devoted to study the qualitative properties exhibited by the segregated states belonging to the class \mathcal{S} . In particular we shall establish the local lipschitz continuity both of U and its nodal set; to this aim we will take advantage of some monotonicity formulae as in [2, 6]. Then, for the dimension $N = 2$, we develop further our investigation. Our main result is that, near a zero point, U and its null set exhibit the same qualitative behavior of harmonic functions and their nodal sets ([1, 21]). In particular we prove that the set of double points (i.e. points where two densities meet)

is the union of a finite number of regular arcs meeting at a finite number of multiple points (i.e. points where more than two densities meet). We emphasize that, at a multiple point the densities share the angle in equal parts and moreover an asymptotic expansion for U is available.

The plan of the paper is the following: in Section 2 we introduce the basic assumptions and formulate the variational problem; the existence of a minimizer is proven in Section 3; Section 4 deals with the uniqueness of the solution; finally in Section 5 the extremality conditions are established. In Sections 6 and 7 we introduce \mathcal{S} and a wider functional class \mathcal{S}^* ; next in Section 8 we prove the local lipschitz continuity in \mathcal{S}^* and the global regularity in \mathcal{S} . In Section 9 we establish some qualitative properties of the elements of \mathcal{S} in dimension $N = 2$.

2 Assumptions and notation

Let $N \geq 2$; let $\Omega \subset \mathbb{R}^N$ be a connected, open bounded domain with regular boundary $\partial\Omega$. Let $k \geq 2$ be a fixed integer. Throughout all the paper we will make the following set of assumptions (for every $i = 1, \dots, k$):

- $\phi_i \in H^{1/2}(\partial\Omega)$, $\phi_i \geq 0$, $\phi_i \cdot \phi_j = 0$ for $i \neq j$, almost everywhere on $\partial\Omega$; sometimes such a boundary datum will be called *admissible*
- $d_i \in W^{2,\infty}(\Omega)$, $d_i > 0$ on $\overline{\Omega}$
- $f_i(x, s) : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$ such that:

(A1) $f_i(x, s)$ is Lipschitz continuous in s , uniformly in x and $f_i(x, 0) \equiv 0$

(A2) there exists $b_i \in L^\infty(\Omega)$ such that both

$$|f_i(x, s)| \leq b_i(x)s \quad \forall x \in \Omega, s \geq \bar{s} > 1$$

and

$$\int_{\Omega} \left(d_i^2(x) |\nabla w(x)|^2 - b_i(x) w^2(x) \right) dx > 0 \quad \forall w \in H_0^1(\Omega).$$

Every ϕ_i will be the boundary trace of a non negative density $u_i \in H^1(\Omega)$. Moreover, associated to each density, we consider its diffusion coefficient d_i and its internal potential $F_i(x, s) := \int_0^s f_i(x, u) du$.

We are concerned with the following variational problem.

Problem 2.1 Let

$$\mathcal{U} = \left\{ (u_1, \dots, u_k) \in (H^1(\Omega))^k : u_i|_{\partial\Omega} = \phi_i, u_i \geq 0 \ \forall i = 1, \dots, k; \ u_j \cdot u_i = 0, i \neq j \text{ a.e. on } \Omega \right\}.$$

Find the minimum of the functional

$$J(U) = \sum_{i=1, \dots, k} \left\{ \int_{\Omega} \left(\frac{1}{2} d_i^2(x) |\nabla u_i(x)|^2 - F_i(x, u_i(x)) \right) dx \right\} \quad (1)$$

where $U \in \mathcal{U}$.

Remark 2.1 We notice that the case with variable diffusions can be reduced, by a suitable change of the variables u_i 's, to the case when $d_i \equiv 1$ for every i . Indeed, let $u_i = v_i/d_i$ (this can be done because the d_i 's are strictly positive). After integrating by parts we obtain

$$\begin{aligned} J(U) &= \sum_{i=1, \dots, k} \left\{ \int_{\Omega} \left(\frac{1}{2} |\nabla v_i|^2 + \frac{1}{2} \frac{\Delta d_i}{d_i} v_i^2 - F_i \left(x, \frac{v_i}{d_i} \right) \right) dx - \frac{1}{2} \int_{\partial\Omega} \phi_i^2 d_i \frac{\partial d_i}{\partial \nu} ds \right\} \\ &= \sum_{i=1, \dots, k} \left\{ \int_{\Omega} \left(\frac{1}{2} |\nabla v_i|^2 - \widetilde{F}_i(x, v_i) \right) dx - C \right\} = \widetilde{J}(V) - C. \end{aligned}$$

Exploiting this identity, the reader can easily check that the validity of the assumptions for the f_i 's and for the ϕ 's implies the same for the new data \widetilde{f}_i 's and $\widetilde{\phi}_i$'s. Hence, in what follows, we will choose $d_i \equiv 1$ for every i , unless otherwise specified (namely in the results of Section 4).

Remark 2.2 By our definition, the functions f_i 's are defined only for non negative values of s (recall that our densities u_i 's are assumed non negative); thus we can arbitrarily define such functions on the negative semiaxis. For the sake of convenience, when $s \leq 0$ we will let $f_i(x, s) := -f_i(x, -s)$. This extension preserves the continuity, thanks to assumption (A1). In the same way, each F_i is extended as an even function.

Notation In the following, when not needed, we shall omit the dependence on the variable x . We use the standard notation $g^+(x) = \max_{x \in \Omega} (g(x), 0)$ and $g^-(x) = \max_{x \in \Omega} (-g(x), 0)$. Given a k -uple (u_1, \dots, u_k) we introduce the "hat" operation as

$$\widehat{u}_i := u_i - \sum_{j \neq i} u_j.$$

Furthermore, with some abuse of notation, we shall use a capital letter to identify both a k -uple and the sum of its k components (e.g. $U = (u_1, \dots, u_k)$ and $U = \sum_{i=1}^k u_i$). With the notation $(u_{i,n})$ we shall denote the i -th component of a sequence of k -uples (U_n) . The symbol χ_A will denote the characteristic function of the set A .

3 Existence of the minimum and continuous dependence

Our first goal is to prove the existence of at least one minimizer of Problem 2.1. Next we shall prove continuous dependence of the minimizers (that do not need to be unique) with respect to the data. This continuity property will be exploited in the analysis of the local properties of the solutions, and precisely when performing the blow-up argument.

To this aim, we start observing that our assumptions on f_i imply

$$|F_i(x, s)| \leq \frac{b_i(x)}{2} s^2 + C|s| \quad \forall x \in \Omega, \forall s \in \mathbb{R} \quad (2)$$

for every i . On the other hand, by standard eigenvalues theory, assumptions (A2) implies that the quadratic form there is an equivalent norm on $H_0^1(\Omega)$, that is, there exists $\varepsilon > 0$ such that

$$\int_{\Omega} (|\nabla w|^2 - b_i(x)w^2) dx \geq \varepsilon \int_{\Omega} |\nabla w|^2 dx,$$

for every $w \in H_0^1(\Omega)$ and for every i . As a consequence we have the following result:

Theorem 3.1 *Under the assumptions of Section 2 Problem 2.1 has at least one solution.*

Proof: applying (2) we easily obtain that the H^1 -continuous functional (1) is coercive; indeed, for every $u_i \in H^1(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} \left(\frac{1}{2} |\nabla u_i|^2 - F_i(u_i) \right) dx &\geq \int_{\Omega} \frac{1}{2} (|\nabla u_i|^2 - b_i u_i^2) dx - \int_{\Omega} C u_i dx \\ &\geq \frac{\varepsilon}{2} \int_{\Omega} |\nabla u_i|^2 - c(\Omega) \end{aligned}$$

for some constant $c(\Omega)$. Let us take a minimizing sequence $(u_{i,n})$ in \mathcal{U} of disjoint support functions. This sequence being H^1 -bounded by the above inequality, there exists a subsequence weakly convergent to u_i in H^1 and, by compact injection, in the L^2 -strong topology; taking possibly a new subsequence, we infer almost everywhere convergence in Ω of every $u_{i,n}$ to u_i and the limit functions have obviously disjoint supports. The weak lower semicontinuity ensures that the weak limit is in fact a minimizer. ■

Theorem 3.2 *Let $U_n = (u_{1,n}, \dots, u_{k,n})$ be a sequence of solutions to Problem 2.1, respectively with admissible data $(\phi_{1,n}, \dots, \phi_{k,n})$, such that*

$$\phi_{i,n} \longrightarrow \phi_i \quad \text{in } H^{1/2}(\partial\Omega)$$

and potentials

$$F_{i,n}(x, s) \longrightarrow F_i(x, s) \quad \text{in } C^1(\Omega \times \mathbb{R})$$

for every $i = 1, \dots, k$. Then,

$$u_{i,n} \longrightarrow u_i \quad \text{in } H^1(\Omega)$$

for every $i = 1, \dots, k$ and $U = (u_1, \dots, u_k)$ solves Problem 2.1 with data (ϕ_1, \dots, ϕ_k) and potentials (F_1, \dots, F_k) .

Proof: first, we observe that (ϕ_1, \dots, ϕ_k) is an admissible datum, i.e. the ϕ_j 's are nonnegative and have disjoint supports by the strong convergence of $(\phi_{1,n}, \dots, \phi_{k,n})$ in $H^{1/2}(\partial\Omega)$.

We denote by $U^* = (u_1^*, \dots, u_k^*)$ a solution to Problem 2.1 with data (ϕ_1, \dots, ϕ_k) , and (F_1, \dots, F_k) . Consider the minimum levels

$$c_n = J(U_n) \quad \text{and} \quad c^* = J(U^*).$$

Observe that the convergence of the boundary traces $\phi_{i,n}$'s and of the $F_{i,n}$'s, ensures a bound on the sequence c_n . The coercivity of J then yields to a bound on the sequence $\|u_{i,n}\|_{H^1(\Omega)}$; therefore we can assume, up to a subsequence, that

$$\begin{aligned} c_n &\longrightarrow c^0 \\ u_{i,n} &\longrightarrow u_i \text{ weakly in } H^1(\Omega). \end{aligned} \tag{3}$$

Furthermore, as a consequence of the compact injection $H^1 \hookrightarrow L^2$, we have $u_i \cdot u_j = 0$, whenever $i \neq j$. Moreover, by the weak continuity of the trace operator, we obtain

$$u_i|_{\partial\Omega} = \phi_i.$$

The lower weak semicontinuity of the norm implies

$$\frac{1}{2} \sum_{i=1, \dots, k} \int_{\Omega} |\nabla u_i|^2 \leq \liminf_{n \rightarrow \infty} \frac{1}{2} \sum_{i=1 \dots k} \int_{\Omega} |\nabla u_{i,n}|^2$$

and we also have

$$\sum_{i=1, \dots, k} \int_{\Omega} F_i(x, u_i(x)) = \lim_{n \rightarrow \infty} \sum_{i=1, \dots, k} \int_{\Omega} F_{i,n}(x, u_{i,n}(x)).$$

We observe that the level

$$c = \sum_{i=1 \dots k} \left\{ \frac{1}{2} \int_{\Omega} |\nabla u_i|^2 - \int_{\Omega} F_i(u_i) \right\}$$

is not necessary a minimum level but satisfies, by the discussion above, the inequalities

$$c_0 \geq c \geq c^*.$$

We wish to prove that $c_0 = c^*$. Suppose, by contradiction, that $c^* < c_0$. Consider the harmonic extensions (still denoted with the same symbols) on Ω of the $\phi_{i,n}$'s and of the ϕ_i 's and introduce $\psi_{i,n} = \phi_{i,n} - \phi_i$. Then, by construction

$$\begin{aligned} \psi_{i,n} &\longrightarrow 0 \text{ in } H^1(\Omega) \\ \psi_{i,n}|_{\partial\Omega} &\longrightarrow 0 \text{ in } H^{1/2}(\partial\Omega). \end{aligned} \tag{4}$$

We define

$$\begin{aligned} w_{i,n} &= (u_i^* + \psi_{i,n})^+ \\ v_{i,n} &= (w_{i,n} - \sum_{j \neq i} w_{j,n})^+. \end{aligned}$$

We observe that $w_{i,n}|_{\partial\Omega} = \phi_{i,n}$; moreover, since $u_i^* \geq 0$ and by (4),

$$\begin{aligned} w_{i,n} &\longrightarrow u_i^* \text{ in } H^1(\Omega) \\ v_{i,n} &\longrightarrow u_i^* \text{ in } H^1(\Omega). \end{aligned} \tag{5}$$

Moreover, since $w_{i,n} \geq 0$, it is immediate to see that $v_{i,n} \cdot v_{j,n} = 0$ if $i \neq j$. Hence it follows from the definition of c_n that

$$\sum_{i=1, \dots, k} \left\{ \frac{1}{2} \int_{\Omega} |\nabla v_{i,n}|^2 - \int_{\Omega} F_{i,n}(v_{i,n}) \right\} \geq c_n ;$$

on the other hand (5) implies

$$\sum_{i=1, \dots, k} \left\{ \frac{1}{2} \int_{\Omega} |\nabla v_{i,n}|^2 - \int_{\Omega} F_{i,n}(v_{i,n}) \right\} \longrightarrow c^*$$

that implies that $c^* \geq c_0$.

Finally, from the equality of the minima $c_0 = c = c^*$, we also deduce the strong convergence of the $u_{i,n}$'s in $H^1(\Omega)$ and the thesis follows. \blacksquare

4 Uniqueness

In general we can not expect the minimizer of Problem 2.1 to be unique. A simple counterexample can be constructed in the following way:

Example 4.1 Let $k = 2$, $\phi_1 \equiv \phi_2 \equiv 0$, $f_1(x, s) = f_2(x, s) = \min(\lambda s, s^{1/3})$, where $\lambda > \lambda_1(\Omega)$, the first eigenvalue of the Laplace operator with Dirichlet boundary condition. The reader can easily check that the assumptions of Theorem 3.1 are fulfilled and that the infimum is smaller than zero. Moreover the infimum of the associated functional is achieved by a function of the form $(u, 0)$, or $(0, u)$, where $u \not\equiv 0$; indeed, any minimizer (u_1, u_2) can be replaced by, say, $(u_1 + u_2, 0)$ keeping the same level of the functional. Hence the associated variational problem does not have uniqueness of solutions.

A major obstruction to uniqueness is the lack of convexity that may occur both in the Lagrangian (as the above example illustrates) and in the constraint; nevertheless, the following result shows that the full convexity of the Lagrangian is sufficient to prove uniqueness of the minimizer, provided the diffusions do not depend on i .

Theorem 4.1 Under the assumptions of Section 2, assume moreover that

$$d_i \equiv d_j, \forall i, j \quad (6)$$

$$F_i(x, s) \text{ is concave in the variable } s, \text{ for all } x \in \Omega \quad (7)$$

Then Problem 2.1 has an unique minimizer.

Proof: let $c = \inf\{J(U) : U \in \mathcal{U}\}$. Arguing by contradiction, we consider two minimizers $U = (u_1, \dots, u_k)$ and $V = (v_1, \dots, v_k)$ achieving c , with $u_i \not\equiv v_i$ for some i . For every i and $\lambda \in [0, 1]$, we define

$$\hat{u}_i = u_i - \sum_{h \neq i} u_h \quad (8)$$

$$\hat{v}_i = v_i - \sum_{h \neq i} v_h \quad (9)$$

$$w_i^{(\lambda)} = [\lambda \hat{u}_i + (1 - \lambda) \hat{v}_i]^+.$$

Our goal is to show that $J(w_i^{(\lambda)}) < \lambda J(u_i) + (1 - \lambda) J(v_i)$, for every $\lambda \in (0, 1)$. It is worthwhile noticing that this property can be seen as a convexity type property combined with a special type of projection on the constraint \mathcal{U} .

To begin with, we have to show that the $w_i^{(\lambda)}$'s satisfy the constraint. We first notice that $w_i^{(\lambda)} \geq 0$ and $w_i|_{\partial\Omega} = \phi_i$. Furthermore we have

$$w_i^{(\lambda)} w_j^{(\lambda)} = 0 \text{ a.e. on } \Omega, \text{ for } j \neq i.$$

Indeed, assume $w_i^{(\lambda)}(x) > 0$; this means that

$$\lambda u_i(x) + (1 - \lambda) v_i(x) > \sum_{h \neq i} \lambda u_h(x) + (1 - \lambda) v_h(x) \geq \lambda u_j(x) + (1 - \lambda) v_j(x) \quad \forall j \neq i.$$

Therefore we have, when $j \neq i$,

$$\lambda u_j(x) + (1 - \lambda)v_j(x) < \lambda u_i(x) + (1 - \lambda)v_i(x) \leq \sum_{h \neq j} \lambda u_h(x) + (1 - \lambda)v_h(x) ,$$

and hence $w_j^{(\lambda)}(x) \leq 0$.

Let us denote the supports

$$\Gamma_i^{(\lambda)} = \{w_i^{(\lambda)} > 0\} ;$$

recalling that $F_i(x, 0) \equiv 0$ we note that

$$J\left(\sum_{i=1}^k w_i^{(\lambda)}\right) = \sum_{i=1}^k \int_{\Gamma_i^{(\lambda)}} \left(\frac{1}{2}d^2|\nabla w_i^{(\lambda)}|^2 - F_i(w_i^{(\lambda)})\right)dx.$$

In view of (6), using the convexity of the quadratic part of the functional, and the definitions (8),(9) and keeping in mind that both the $(u_i)'$ s and $(v_i)'$ s have disjoint supports, we obtain that, for every $\lambda \in (0, 1)$,

$$\begin{aligned} \sum_{i=1}^k \int_{\Gamma_i^{(\lambda)}} \left(\frac{1}{2}d^2|\nabla w_i^{(\lambda)}|^2\right)dx &< \sum_{i=1}^k \int_{\Gamma_i^{(\lambda)}} \lambda\left(\frac{1}{2}d^2|\nabla \hat{u}_i|^2\right) + (1 - \lambda)\left(\frac{1}{2}d^2|\nabla \hat{v}_i|^2\right)dx \\ &\leq \sum_{i=1}^k \int_{\Omega} \lambda\left(\frac{1}{2}d^2|\nabla u_i|^2\right) + (1 - \lambda)\left(\frac{1}{2}d^2|\nabla v_i|^2\right)dx \end{aligned}$$

Now we turn to the potential integral. By assumption (7) and the evenness of the potentials F_i 's, the inequality

$$-F_i(x, \lambda \hat{u}_i(x) + (1 - \lambda)\hat{v}_i(x)) \leq -\lambda F_i(x, \hat{u}_i(x)) - (1 - \lambda)F_i(x, \hat{v}_i(x))$$

holds whenever $x \in \{u_i > 0\} \cap \{v_i > 0\}$. Hence, assume that $x \in \{u_i > 0\} \cap \{v_j > 0\}$, for some $j \neq i$; let us fix λ and let x such that $\lambda u_i(x) - (1 - \lambda)v_j(x) > 0$ (the symmetric case is obtained by parity), and we study the term $-F_i(x, \lambda u_i(x) - (1 - \lambda)v_j(x))$. Introducing the auxiliary function $\Psi(\lambda) = -F_i(x, \lambda u_i - (1 - \lambda)v_j) + \lambda F_i(x, u_i) + (1 - \lambda)F_j(x, v_j)$, let $\bar{\lambda} \in (0, 1)$ such that $\bar{\lambda} u_i - (1 - \bar{\lambda})v_j = 0$. It is easy to see that $\Psi(1) = 0$, and $\Psi''(\lambda) > 0$. Moreover, $\Psi(\bar{\lambda}) \leq 0$, since $F_i(x, 0) \equiv 0$ and, by (7), $F_i \leq 0$. Hence, by convexity, we infer that $\Psi(\lambda) < 0$, for every $\lambda \in (\bar{\lambda}, 1)$ and therefore

$$-F_i(x, (\lambda \hat{u}_i + (1 - \lambda)\hat{v}_j)^+) \leq -\lambda F_i(x, u_i) - (1 - \lambda)F_j(x, v_j)$$

holds when $x \in \{u_i > 0\} \cap \{v_j > 0\}$.

Finally, gathering together all these inequalities, for every fixed λ we obtain

$$J\left(\sum_{i=1}^k w_i^{(\lambda)}\right) < \lambda J\left(\sum_{i=1}^k u_i\right) + (1 - \lambda)J\left(\sum_{i=1}^k v_i\right) < c ,$$

a contradiction. ■

The requirement of Theorem 4.1 that the Lagrangians are convex in all variables may seem very restrictive. The following result makes a different assumption, still sufficient for the uniqueness, that may be more useful in the applications, for it is always satisfied (given the d_i 's and F_i 's) provided the domain Ω is small enough.

Corollary 4.1 *Let $d_i \in W^{2,\infty}$ be given. Assume that F_i are of class \mathcal{C}^2 in the variable s and let us denote*

$$b_i(x) = \sup_{s \in \mathbb{R}} \frac{\partial^2 F_i}{\partial s^2}(x, s) \quad (10)$$

Assume that there exists a positive function d such that, for every $i = 1, \dots, k$,

$$-\Delta d + \left(\frac{\Delta d_i}{d_i} - \frac{b_i}{2d_i^2} \right) d \geq 0 \quad \text{in } \Omega. \quad (11)$$

Then, Problem 2.1 has a unique minimizer.

Proof: the following identities hold:

$$\begin{aligned} \int_{\Omega} d_i^2(x) |\nabla u_i|^2 &= \int_{\Omega} d^2(x) \left| \nabla \left(\frac{d_i u_i}{d} \right) \right|^2 - d^2 \nabla \left(\frac{d_i}{d} \right) \cdot \nabla \left(\frac{d_i u_i^2}{d} \right) \\ &= \int_{\Omega} d^2(x) \left| \nabla \left(\frac{d_i u_i}{d} \right) \right|^2 + \operatorname{div} \left(d^2 \nabla \left(\frac{d_i}{d} \right) \right) \frac{d_i u_i^2}{d} - \int_{\partial \Omega} d d_i u_i^2 \nabla \left(\frac{d_i}{d} \right) \cdot \nu \end{aligned}$$

Therefore, up to a constant (recall that our boundary data are prescribed), by the change of variables $v_i = d_i u_i / d$ we can transform the initial Lagrangians into ones of the following form:

$$d^2(x) |\nabla v_i|^2 + \frac{d}{d_i} \operatorname{div} \left(d^2 \nabla \left(\frac{d_i}{d} \right) \right) v_i^2 - F_i(x, d v_i / d_i).$$

Recalling the definition of the b_i 's of (10), one easily checks that these Lagrangians are convex in the variable v_i provided the following inequality holds for every index i

$$2 \frac{d}{d_i} \operatorname{div} \left(d^2 \nabla \left(\frac{d_i}{d} \right) \right) - \frac{d^2}{d_i^2} b_i \geq 0$$

that is easily seen to be equivalent to (11). ■

5 Extremality conditions

The goal of this section is to prove that the minimizers of the variational problem Problem 2.1 satisfy a suitable set of differential inequalities. To start with, let (u_1, \dots, u_k) be a minimizer of problem 2.1; we define $\hat{f}(x, \hat{u}_i)$

$$\hat{f}(x, \hat{u}_i) = \sum_j f_j(x, \hat{u}_i) \chi_{\{u_j > 0\}} = \begin{cases} f_i(u_i) & \text{if } x \in \{u_i > 0\} \\ -f_j(x, u_j) & \text{if } x \in \{u_j > 0\}, j \neq i. \end{cases} \quad (12)$$

Note that this definition is consistent with that of (8), for the functions f_i are extended by oddness. Our main result is the following:

Theorem 5.1 *Let U be a solution to Problem 2.1. Then, for every i , we have, in distributional sense,*

$$(i) \quad -\Delta u_i \leq f_i(x, u_i)$$

$$(ii) \quad -\Delta \hat{u}_i \geq \hat{f}(x, \hat{u}_i) .$$

Proof:

- (i) We argue by contradiction. Then, there exists at least one index j such that the claim does not hold; that is, there is $0 \leq \phi \in C_c^\infty(\Omega)$ such that

$$\int_{\Omega} \nabla u_j \nabla \phi - f_j(x, u_j) \phi > 0.$$

For $0 < t < 1$ we define a new test function $V = (v_1, \dots, v_k)$ as follows:

$$v_i = \begin{cases} u_i & \text{if } i \neq j \\ (u_i - t\phi)^+ & \text{if } i = j. \end{cases}$$

We claim that V lowers the value of (1); indeed we have

$$\begin{aligned} J(V) - J(U) &= \int_{\Omega} \frac{1}{2} (|\nabla(u_j - t\phi)^+|^2 - |\nabla u_j|^2) - \int_{\Omega} F_j(x, (u_j - t\phi)^+) - F_j(x, u_j) \\ &\leq \int_{\Omega} \frac{1}{2} (|\nabla u_j - t\phi|^2 - |\nabla u_j|^2) + t \int_{\Omega} f_j(x, u_j) \phi + o(t) \\ &\leq t \int_{\Omega} (-\nabla u_j \nabla \phi + f_j(x, u_j) \phi) + o(t). \end{aligned}$$

Choosing t sufficiently small, we obtain

$$J(V) - J(U) < 0,$$

a contradiction.

- (ii) Let j and $0 < \phi \in C_c^\infty(\Omega)$ such that

$$\int_{\Omega} \nabla \hat{u}_j \nabla \phi - \hat{f}(x, \hat{u}_j) \phi dx < 0.$$

Again, we show that the value of the functional can be lessened by replacing U with an appropriate new test function V . To this aim we consider the positive and negative parts of $\hat{u}_j + t\phi$ and we notice that, obviously,

$$\{(\hat{u}_j + t\phi)^- > 0\} \subset \{(\hat{u}_j)^- > 0\} = \cup_{i \neq j} \{u_i > 0\} .$$

Let us define $V = (v_1, \dots, v_k)$ in the following way:

$$v_i = \begin{cases} (\hat{u}_j + t\phi)^+, & \text{if } i = j \\ (\hat{u}_j + t\phi)^- \chi_{\{u_i > 0\}}, & \text{if } i \neq j. \end{cases}$$

We compute, using the definition (12),

$$\begin{aligned} J(V) - J(U) &= \sum_{i=1}^k \int_{\Omega} \frac{1}{2} (|\nabla v_i|^2 - |\nabla u_i|^2) - (F_i(x, v_i) - F_i(x, u_i)) \\ &= \int_{\Omega} \frac{1}{2} (|\nabla \hat{u}_j + t\phi|^2 - |\nabla \hat{u}_j|^2) - (F_j(x, (\hat{u}_j + t\phi)^+) - F_j(x, u_j)) \\ &\quad - \sum_{i \neq j} (F_i(x, (\hat{u}_j + t\phi)^- \chi_{\{u_i > 0\}}) - F_i(x, u_i)) \\ &= t \int_{\Omega} \nabla \hat{u}_j \nabla \phi - \int_{\Omega_j} f_j(x, u_j) \chi_{\{u_j > 0\}} \phi + \sum_{i \neq j} f_i(x, u_i) \chi_{\{u_i > 0\}} \phi + o(t) \\ &= t \int_{\Omega} \nabla \hat{u}_j \nabla \phi - \int_{\Omega} \hat{f}_j(x, \hat{u}_j) \phi + o(t). \end{aligned}$$

For t small enough we find $J(V) < J(U)$, a contradiction. ■

6 The class \mathcal{S} and its basic properties

Let \mathcal{U} be the set of admissible k -uples as defined in Problem 2.1 and let f_i be given satisfying (A1), (A2); we introduce the following functional class:

Definition 6.1

$$\mathcal{S} = \left\{ (u_1, \dots, u_k) \in \mathcal{U} : -\Delta u_i \leq f_i(x, u_i), -\Delta \hat{u}_i \geq \hat{f}(x, \hat{u}_i), \forall i = 1, \dots, k \right\}$$

By virtue of Theorem 5.1 the class \mathcal{S} is the natural framework where to develop our theory of regularity and free boundary of minimizers to Problem 2.1. As already noticed in the introduction, the class \mathcal{S} is of independent interest, for it contains the asymptotic limits of highly competing diffusion systems.

Let us start with the following definitions:

Definition 6.2 The multiplicity of a point $x \in \Omega$ is

$$m(x) = \sharp \{i : meas(\{u_i > 0\} \cap B(x, r)) > 0 \forall r > 0\}.$$

We shall denote by

$$\mathcal{Z}_h(U) = \{x \in \Omega : m(x) \geq h\}$$

the set of points of multiplicity greater or equal than $h \in \mathbb{N}$.

The following properties are straightforward consequences of the definition of \mathcal{S} joint to the locally Lipschitz continuity of the f_i 's that implies the validity of the Maximum Principle for elliptic equations.

Proposition 6.1 *Let $x \in \Omega$:*

- (a) *If $m(x) = 0$, then there is $r > 0$ such that $u_i \equiv 0$ on $B(x, r)$, for every i .*
- (b) *If $m(x) = 1$, then there are i and $r > 0$ such that $u_i > 0$ and $-\Delta u_i = f_i(x, u_i)$ on $B(x, r)$.*
- (c) *If $m(x) = 2$, then are i, j and $r > 0$ such that $u_k \equiv 0$ for $k \neq i, j$ and $-\Delta(u_i - u_j) = f_{i,j}(x, u_i - u_j)$ on $B(x, r)$, where $f_{i,j}(x, s) = f_i(x, s^+) - f_j(x, s^-)$.*

Proof: part (a) follows directly by the definition of multiplicity; assume $m(x) = 1$ and let $r > 0$ be such that $\text{meas}(\{u_j > 0\} \cap B(x, r)) = 0$ for every j except i . Then, by definition of \mathcal{S} , u_i satisfies the equation $-\Delta u_i = f_i(x, u_i)$ on $B(x, r)$. We can write

$$-\Delta u_i = a_i(x)u_i = \frac{f_i(x, u_i)}{u_i(x)}u_i$$

where $a_i \in L^\infty$ by the Lipschitz continuity of the f_i . Then, since $u_i \geq 0$, we infer from the strong maximum principle that $u_i > 0$ on $B(x, r)$. The second statement follows immediately from the definition. ■

Remark 6.1

- We can not exclude, at this stage, the occurrence of points of multiplicity zero, although this possibility will be ruled out at the end of Section 9.2, at least in two dimensions, under a weak non degeneracy assumption. Note that $\partial\{x \in \Omega : m(x) = 0\} \subset \mathcal{Z}_3 \cup \partial\Omega$.
- The second point of Proposition 6.1 says in particular that, if $m(x) = 2$

$$\lim_{\substack{y \rightarrow x \\ y \in \{u_i > 0\}}} \nabla u_i(y) = - \lim_{\substack{y \rightarrow x \\ y \in \{u_j > 0\}}} \nabla u_j(y) .$$

If, by the way, the above limit is not zero, it follows that the set $\{x : m(x) = 2\}$ is locally a \mathcal{C}^1 manifold of dimension $N - 1$. Of course the above equality has to be changed, in presence of variable diffusions, into

$$\lim_{\substack{y \rightarrow x \\ y \in \{u_i > 0\}}} d_i(x) \nabla u_i(y) = - \lim_{\substack{y \rightarrow x \\ y \in \{u_j > 0\}}} d_j(x) \nabla u_j(y) .$$

A major goal in the subsequent analysis will concern the geometrical properties of the supports of the densities u_i and their common boundaries. As a first consequence of the Maximum Principle we can give a criterium for the connectedness of the supports:

Proposition 6.2 *Assume, for some i , that u_i is continuous and $\{u_i > 0\} \cap \partial\Omega$ is connected. If*

$$\frac{f_i(x, s)}{s} < \lambda_1(\Omega) \tag{13}$$

where λ_1 denotes the first eigenvalue of the Dirichlet operator with zero boundary condition, then $\{u_i > 0\}$ is connected.

Proof: assume not, then there is a connected component A of $\{u_i > 0\}$ that does not touch the boundary; on the other hand, $u_i > 0$ satisfies the equation

$$-\Delta u_i = \left(\frac{f_i(u_i)}{u_i} \right) u_i$$

on A with vanishing boundary trace. When testing the equation with u_i , by (13) we obtain a contradiction with the Poincaré inequality. \blacksquare

It is worthwhile noticing that the condition (13) is always satisfied on small domains. This can be useful in the local analysis of the solutions. Another useful property of \mathcal{S} is that its elements are uniformly bounded in the interior of Ω thanks to the next proposition.

Proposition 6.3 *Let (u_i) be an element of \mathcal{S} . Then the following hold*

(i) *There are functions $(\Phi_i) \in W_{\text{loc}}^{1,\infty}(\Omega)$ such that*

$$\begin{aligned} -\Delta \Phi_i &= f_i(x, \Phi_i) && \text{in } \Omega \\ \Phi_i &= u_i && \text{on } \partial\Omega \\ \Phi_i &\geq u_i && \text{in } \Omega \end{aligned} \tag{14}$$

(ii) *There are functions $(\Psi_i) \in W_{\text{loc}}^{1,\infty}(\Omega)$ such that*

$$\begin{aligned} -\Delta \Psi_i &= \hat{f}(x, \hat{u}_i) && \text{in } \Omega \\ \Psi_i &= \hat{u}_i && \text{on } \partial\Omega \\ \Psi_i &\leq \hat{u}_i && \text{in } \Omega \end{aligned} \tag{15}$$

(in particular, $\Psi_i^+ \leq \hat{u}_i^+ = u_i$).

Proof: to prove the first assertion we apply the method of upper-lower solutions: we need an ordered pair of functions $\alpha_i \leq \beta_i$ where α_i is subsolution and β_i supersolution of problem (14). We simply let $\alpha_i = u_i$ as lower solution; on the other hand we obtain a suitable β_i by solving

$$\begin{aligned} -\Delta \beta_i &= b_i(x) \beta_i && \text{in } \Omega \\ \beta_i &= M + \phi_i && \text{on } \partial\Omega \end{aligned}$$

for large constants M . Notice that assumption (A2) implies the existence of arbitrarily large positive functions β_i satisfying the above problem. Furthermore, since $b(x)\beta_i \geq f_i(x, \beta_i)$, then the β_i are supersolutions to equation (14). Finally we get $\beta_i \geq \alpha_i$ by the maximum principle.

The proof of the second assertion is trivial, since the boundedness of \hat{u}_i and the assumption (A1) imply the existence of a solution for problem (15); the relation $\Psi_i \leq \hat{u}_i$ then follows by the maximum principle.

To conclude we observe that the regularity of the Φ_i 's and Ψ_i 's follows by the standard regularity theory for elliptic equations and our assumptions on the boundary data and the nonlinearities. \blacksquare

Remark 6.2 As a consequence of the above proposition, the components of each element $U \in \mathcal{S}$ are uniformly bounded on compact subsets of Ω . Then, recalling (A1),(A2), if $\omega \subset\subset \Omega$ there exists $M > 0$ (depending only on ω) such that

$$U \in \mathcal{S} \implies -\Delta u_i \leq M, \quad -\Delta \hat{u}_i \geq -M \quad \text{on } \omega \quad (16)$$

Furthermore, the regularity can be improved up to the boundary of Ω in the sense that to bounded boundary data there correspond bounded barriers Φ_i and Ψ_i . Moreover the barriers will be Lipschitz continuous up to the boundary when both the data and the boundary $\partial\Omega$ enjoy the same regularity.

7 The class \mathcal{S}^*

Let $M \geq 0$ and h be a fixed integer. We introduce

Definition 7.1

$$\mathcal{S}_{M,h}^*(\omega) = \left\{ (u_1, \dots, u_h) \in (H^1(\omega))^h : \begin{array}{l} u_i \geq 0, \ u_i \cdot u_j = 0 \text{ if } i \neq j \\ -\Delta u_i \leq M, \ -\Delta \hat{u}_i \geq -M \end{array} \right\}$$

It follows from Remark 6.2, that $\mathcal{S} \subset \mathcal{S}_{M,h}^*$. It will be more convenient to work in this larger class rather than in the class \mathcal{S} , for it is closed with respect to the limits of sequences of scaled functions. This property will be extremely useful in performing the blow-up analysis in Section 8. We first present a technical result concerning with the elements of $\mathcal{S}_{M,h}^*(B(0,1))$ when 0 is a point of multiplicity at least two.

Proposition 7.1 *There exists M^* such that, for all $0 < M < M^*$ the following holds*

(a) *Let $U_n \in \mathcal{S}_{M,h}^*(B(0,1))$ such that $m(U_n)(0) \geq 2$ and $U_n \rightharpoonup U$ with $U \not\equiv 0$. Then*

$$\#\{i = 1, \dots, h : u_i \not\equiv 0\} \geq 2.$$

(b) *For every $\gamma > 0$ there exists $C(\gamma) > 0$ such that, if $U \in \mathcal{S}_{M,h}^*(B(0,1))$ with $m(U)(0) \geq 2$ and $\|U\|_{L^2(B(0,1))} \geq \gamma$, then*

$$\int_{B(0,1)} |\nabla U|^2 \geq C(\gamma) \int_{B(0,1)} |U|^2$$

where $C > 0$ only depends on M^ and γ .*

(c) *Let $U_n \in \mathcal{S}_{M,h}^*(B(0,1))$ such that $m(U_n)(0) \geq 2$; if $\|\nabla U_n\|_{L^2(B(0,1))} = 1$ and $\|\nabla U_n\|_{L^2(\partial B(0,1))}$ is bounded, then there exists $U \not\equiv 0$ such that $U_n \rightharpoonup U$.*

Proof: *Part (a).* Arguing by contradiction we can assume the existence of $M_n \rightarrow 0$ and $U_n \in \mathcal{S}_{M_n, h}^*(B(0, 1))$ such that $m(U_n)(0) \geq 2$, $U_n \rightharpoonup U$ with $U \not\equiv 0$ and $N := \#\{i = 1, \dots, h : u_i \not\equiv 0\} \leq 1$. Note that, since $U \not\equiv 0$, then N must be equal to 1. Thus we can assume that U is of the form $U = (u_1, 0, \dots, 0)$ where $u_1 \not\equiv 0$. Now consider $\phi > 0$ solution of the problem $-\Delta\phi = 1$ on $B(0, 1)$ with boundary conditions $\phi = 0$. It holds

$$-\Delta(\hat{u}_{1,n} + M_n\phi) \geq 0$$

and by the mean value property for superharmonic functions we have

$$0 = \hat{u}_{1,n}(0) \geq \int_{B(0,1)} \hat{u}_{1,n}(x) dx + M_n \left(\int_{B(0,1)} \phi(x) dx - \phi(0) \right). \quad (17)$$

By the compact embedding of H^1 in L^1 and since $\hat{u}_1 = u_1$, it holds

$$\int_{B(0,1)} \hat{u}_{1,n}(x) dx \rightarrow \int_{B(0,1)} u_1(x) dx = \alpha > 0.$$

On the other hand, the last term of (17) becomes less than $\alpha/2$ when $M_n \rightarrow 0$: this finally provides the contradiction $0 > \alpha/2$.

Part (b). Let $\gamma > 0$ be fixed. Assume by contradiction the existence of $M_n \rightarrow 0$ and $U_n \in \mathcal{S}_{M_n, h}^*(B(0, 1))$ such that $m(U_n)(0) \geq 2$, $\|U_n\|_{L^2(B(0,1))} \geq \gamma$ and

$$\frac{\int_{B(0,1)} |\nabla U_n|^2}{\int_{B(0,1)} |U_n|^2} \rightarrow 0.$$

Let us define $V_n := U_n \cdot \|U_n\|_{L^2(B(0,1))}^{-1}$: note that $V_n \in \mathcal{S}_{M_n/\gamma, h}^*(B(0, 1))$, $m(V_n)(0) \geq 2$; furthermore $\|V_n\|_{L^2(B(0,1))} = 1$ and $\|\nabla V_n\|_{L^2(B(0,1))} \rightarrow 0$ by construction. Then there exists V such that $V_n \rightharpoonup V$; moreover $V \not\equiv 0$, since by the compact embedding of H^1 in L^2 it holds $\|V\|_{L^2(B(0,1))} = 1$. Now note that, since the gradients vanish in the L^2 norm, then $V = (c_1, \dots, c_h)$ where $c_i \in \mathbb{R}$. Furthermore, since the supports of the components of V_n are mutually disjoint, passing to the limit a.e. in $B(0, 1)$, we get $c_i \cdot c_j = 0$ if $i \neq j$: this means that only one of the components of V is not identically 0. This is in contradiction with Part (a) when applied to the sequence V_n .

Part (c). By assumption $\|\nabla U_n\|_{L^2(B(0,1))}$ is bounded: then, if M is small enough we can apply Part (b) and thus the whole H^1 -norm $\|U\|_{H^1(B(0,1))}$ is bounded. This provides U such that $U_n \rightharpoonup U$. Now test the variational inequality $-\Delta u_{i,n} \leq M$ with $u_{i,n}$:

$$\int_{B(0,1)} |\nabla u_{i,n}|^2 \leq M \int_{B(0,1)} u_{i,n} + \int_{\partial B(0,1)} u_{i,n} \frac{\partial}{\partial \nu} u_{i,n}.$$

Assume by contradiction that $U \equiv 0$: by the compact embedding of H^1 in L^p and since $\|\nabla U_n\|_{L^2(\partial B(0,1))}$ is bounded, we deduce that the r.h.s vanishes. This implies $\|\nabla U_n\|_{L^2(B(0,1))} \rightarrow 0$, in contradiction with the assumption $\|\nabla U_n\|_{L^2(B(0,1))} = 1$. \blacksquare

Remark 7.1 The argument used in the proof of part (a) allows to establish a mean value property for functions which are superharmonic up to a small term. Precisely, if $(v_n) \subset H^1$ is such that $-\Delta v_n \geq -M_n$ with $(0 <) M_n \rightarrow 0$ and $\int_{B(0,1)} v_n \geq \alpha > 0$, then $v_n(0) \geq \alpha/2$ if n is large enough.

8 Lipschitz Regularity

A key tool in studying the regularity of both the function U and the free boundary is a suitable version of the celebrated *monotonicity theorem*, see [2, 5]. In this paper we shall take advantage of the following formula which is proven in [6].

Lemma 8.1 *Let $w_1, w_2 \in H^1 \cap L^\infty$ such that $-\Delta w_i \leq 1$, $w_1 \cdot w_2 = 0$ a.e. and $x_0 \in \partial(\{w_i > 0\})$, $i = 1, 2$. Then there exists $C > 0$, independent of x_0 , such that*

$$\prod_{i=1}^2 \frac{1}{r^2} \int_{B(x_0, r)} \frac{|\nabla w_i(x)|^2}{|x - x_0|^{N-2}} dx \leq C. \quad (18)$$

In the proof of our regularity results we shall also need the following technical lemma.

Lemma 8.2 *Let $U \in H^1(\Omega)$ and let us define*

$$\phi(x, r) := \frac{1}{r^N} \int_{B(x, r) \cap \Omega} |\nabla U(y)|^2 dy. \quad (19)$$

If (x_n, r_n) is a sequence in $\overline{\Omega} \times \mathbb{R}^+$ such that

$$\phi(x_n, r_n) \rightarrow \infty$$

then $r_n \rightarrow 0$ and

(i) there exists a sequence $(r'_n)_n \subset \mathbb{R}^+$ such that $\phi(x_n, r'_n) \rightarrow \infty$, and

$$\int_{\partial B(x_n, r'_n) \cap \Omega} |\nabla U|^2 \leq \frac{N}{r'_n} \int_{B(x_n, r'_n) \cap \Omega} |\nabla U|^2;$$

(ii) if $A \subset \overline{\Omega}$ and

$$\frac{\text{dist}(x_n, A)}{r_n} \leq C$$

then there exists a sequence (x'_n, r'_n) such that $\phi(x'_n, r'_n) \rightarrow \infty$ and $x'_n \in A$ for every n .

Proof: since $U \in H^1(\Omega)$ and $\phi(x_n, r_n) \rightarrow +\infty$, obviously $r_n \rightarrow 0$. We begin proving (i). Let g be defined on the whole \mathbb{R}^N as

$$g(x) := \begin{cases} |\nabla U(x)|^2 & x \in \Omega \\ 0 & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Clearly $\int_{B(x, r) \cap \Omega} |\nabla U|^2 = \int_{B(x, r)} g$, $\int_{\partial B(x, r) \cap \Omega} |\nabla U|^2 = \int_{\partial B(x, r)} g$. We observe that

$$\frac{\partial \phi}{\partial r}(x_n, r_n) = \frac{1}{r_n^N} \left(\int_{\partial B(x_n, r_n)} g(y) dy - \frac{N}{r_n} \int_{B(x_n, r_n)} g(y) dy \right).$$

As a consequence, our problem is reduced to find r'_n such that $\phi_r(x_n, r'_n) \leq 0$ and $\phi(x_n, r'_n) \rightarrow +\infty$. Let $r'_n := \inf\{r \geq r_n : \phi_r(x_n, r) \leq 0\}$. We have that $r'_n < +\infty$ for every n (recall that $\phi(x_n, r) \rightarrow 0 < \phi(x_n, r_n)$ as $r \rightarrow \infty$), and $\phi(x_n, r_n) \leq \phi(x_n, r'_n)$, and (i) is proved. In order to prove (ii), let $x'_n \in A$ such that $\text{dist}(x_n, x'_n) \leq 2Cr_n$ (this is possible by assumption) and let $r'_n := (2C+1)r_n$. By construction $B(x'_n, r'_n) \supset B(x_n, r_n)$, that implies $\phi(x'_n, r'_n) \geq (2C+1)^{-N}\phi(x_n, r_n)$, and also (ii) follows. \blacksquare

8.1 Local Lipschitz continuity in \mathcal{S}^*

The class \mathcal{S}^* seems to be the natural framework for proving the interior Lipschitz regularity. Indeed we have:

Theorem 8.1 *Let $M > 0$ and k be a fixed integer. Let $U \in \mathcal{S}_{M,k}^*(\Omega)$: then U is Lipschitz continuous in the interior of Ω .*

Proof: we consider the set $\Omega' \subset \subset \Omega$ compactly enclosed in Ω and the function $\phi(x, r)$ defined as in (19), restricted to the set $D := \{x \in \Omega', r \in \mathbb{R}^+ : 2r < \text{dist}(\partial\Omega', \partial\Omega)\}$. We have to prove that ϕ is bounded on D . We argue by contradiction, assuming that

$$\sup_D \phi(x, r) = +\infty$$

i.e. there exists a sequence (x_n, r_n) , such that

$$\lim_{n \rightarrow \infty} \frac{1}{r_n^N} \int_{B(x_n, r_n)} |\nabla U(y)|^2 dy = +\infty. \quad (20)$$

By Lemma 8.2,(i), there exists a sequence (denoted again by r_n) satisfying (20) and moreover

$$\int_{\partial B(x_n, r_n)} |\nabla U|^2 \leq \frac{N}{r_n} \int_{B(x_n, r_n)} |\nabla U|^2. \quad (21)$$

Now we distinguish three cases, according to the nature of x_n and r_n , up to suitable subsequences:

CASE I: $m(x_n) = 0$ for all n (up to a subsequence) and $\frac{\text{dist}(x_n, \mathcal{Z}_1)}{r_n} \geq 1$.

We immediately obtain a contradiction with (20), since in this case $u_i \equiv 0$ on $B(x_n, r_n)$, for all i . As a consequence, using Lemma 8.2,(ii), without loss of generality, we can assume that $m(x_n) \geq 1$ for every n (besides (20) and (21)).

CASE II: $m(x_n) = 1$ for all n and $\frac{\text{dist}(x_n, \mathcal{Z}_2)}{r_n} \geq 1$.

In this case we can assume that only $u_1 \not\equiv 0$ on $B(x_n, r_n)$ and we define

$$v_n(x) = (u_1(x) - u_1(x_n))^+ \quad w_n(x) = (u_1(x) - u_1(x_n))^-$$

Then v_n and w_n are in $L^\infty(\Omega')$ by Remark 6.1, they have disjoint supports, $-\Delta v_n \leq M$, $-\Delta w_n \leq M$ and $v_n(x_n) = w_n(x) = 0$. Thus we can apply the monotonicity formula of Lemma 8.1

$$\frac{1}{r_n^2} \int_{B(x_n, r_n)} \frac{|\nabla v_n(x)|^2}{|x - x_n|^{N-2}} \cdot \frac{1}{r_n^2} \int_{B(x_n, r_n)} \frac{|\nabla w_n(x)|^2}{|x - x_n|^{N-2}} \leq C$$

uniformly. Since $|x - x_n| < r_n$, we deduce

$$\frac{1}{r_n^N} \int_{B(x_n, r_n)} |\nabla v_n(x)|^2 \cdot \frac{1}{r_n^N} \int_{B(x_n, r_n)} |\nabla w_n(x)|^2 \leq C$$

where C is independent of n . Comparing with (20) we have that only one of the two term is unbounded and forces the second one to vanish, e.g.

$$\frac{1}{r_n^N} \int_{B(x_n, r_n)} |\nabla v_n|^2 \longrightarrow \infty \quad \frac{1}{r_n^N} \int_{B(x_n, r_n)} |\nabla w_n|^2 \longrightarrow 0 \quad (22)$$

as $n \rightarrow \infty$. Let

$$L_n^2 := \frac{1}{r_n^N} \int_{B(x_n, r_n)} |\nabla v_n|^2$$

and let us perform the blow up analysis around x_n with parameter L_n by defining $V_n = (v_{1,n}, v_{2,n})$ as

$$v_{1,n}(x) = \frac{1}{L_n r_n} v_n(x_n + r_n x) \quad v_{2,n}(x) = \frac{1}{L_n r_n} w_n(x_n + r_n x) \quad x \in B(0, 1).$$

It is easy to verify that $V_n \in \mathcal{S}_{M_n, 2}^*(B(0, 1))$, where $M_n = r_n M / L_n \rightarrow 0$. By construction we have that $\int_{B(0, 1)} |\nabla V_n|^2$ is bounded; using (21), this implies that $\int_{\partial B(0, 1)} |\nabla V_n|^2$ is bounded too: thus V_n satisfies the assumptions of Proposition 7.1. This provides the existence of a weak limit $V = (v_1, v_2)$ (by Part (c)) such that $v_i \not\equiv 0$, $i = 1, 2$ (by Part (a)). But this is in contradiction with (22) that forces $v_2 \equiv 0$ (by Part (b)). Again, this contradiction and Lemma 8.2, (ii) allow us to assume $m(x_n) \geq 2$.

CASE III: $m(x_n) \geq 2$ for all n .

In this case the proof follows closely the line of the previous one. Let us give some details: since $m(x_n) \geq 2$, we can apply the monotonicity formula to each pair $w_1 := u_l$, $w_2 := u_k$:

$$\frac{1}{r_n^2} \int_{B(x_n, r_n)} \frac{|\nabla w_1(x)|^2}{|x - x_n|^{N-2}} \cdot \frac{1}{r_n^2} \int_{B(x_n, r_n)} \frac{|\nabla w_2(x)|^2}{|x - x_n|^{N-2}} \leq C$$

uniformly. By (20) we deduce the existence of one index i such that, up to a subsequence, it holds

$$\frac{1}{r_n^N} \int_{B(x_n, r_n)} |\nabla u_i|^2 \longrightarrow \infty \quad \frac{1}{r_n^N} \int_{B(x_n, r_n)} |\nabla u_j|^2 \longrightarrow 0 \quad \forall j \neq i \quad (23)$$

as $n \rightarrow \infty$. Let

$$L_n^2 := \frac{1}{r_n^N} \int_{B(x_n, r_n)} |\nabla u_i|^2$$

and perform the blow up analysis around x_n with parameter L_n by defining $V_n = (v_{j,n})$ as

$$v_{j,n}(x) = \frac{1}{L_n r_n} u_j(x_n + r_n x) \quad x \in B(0, 1).$$

It is easy to verify that $V_n \in \mathcal{S}_{M_n, k}^*(B(0, 1))$, where, again, $M_n = r_n M / L_n \rightarrow 0$. By construction we have that $\int_{B(0, 1)} |\nabla V_n|^2$ is bounded and, again by (21), $\int_{\partial B(0, 1)} |\nabla V_n|^2$ is bounded too: thus V_n satisfies the assumptions of Proposition 7.1. This provides the existence of a weak limit V (by Part (c)) such that at least two of its components are strictly positive (by Part (a)). But this is in contradiction with (23) that forces (by Part (b)) $v_j \equiv 0$ for all $j \neq i$. \blacksquare

8.2 Lipschitz continuity up to the boundary in \mathcal{S}

In this Section we are concerned with the regularity of the elements of \mathcal{S} up to the boundary, in the case of regular boundary and Lipschitz boundary data. Our main goal is the following result.

Theorem 8.2 *Let $\partial\Omega$ be of class C^1 , $U \in \mathcal{S}$ with $u_i|_{\partial\Omega} = \phi_i$ and $\phi_i \in W^{1,\infty}(\partial\Omega)$ for every i . Then $U \in W^{1,\infty}(\overline{\Omega})$.*

The proof relies upon the local analysis as developed in the previous section, joint with the suitable use of the *pinching* property stated in Proposition 6.3. We begin with some preliminary remarks.

Remark 8.1 *Under the assumptions of Theorem 8.2 we have*

- (i) $u_i \in C(\overline{\Omega})$ for every i (and, in particular, it makes sense to consider pointwise values of u_i);
- (ii) $\frac{\partial u_i}{\partial \nu}|_{\partial\Omega} \in L^\infty(\partial\Omega)$ for every i .

Proof: since $U \in \mathcal{S}$, through Proposition 6.3 we obtain the existence of k -uples of functions (Φ_i) , (Ψ_i) , with the properties introduced in that proposition. Moreover, since $\phi_i \in W^{1,\infty}(\partial\Omega)$ for every i , by standard regularity theory for elliptic equations we infer $\Phi_i \in W^{1,\infty}(\overline{\Omega})$, $\Psi_i \in W^{1,\infty}(\overline{\Omega})$ for every i . By Theorem 8.1 (which holds in this case through Remark 6.2) $u_i \in C(\Omega)$; since $\Psi_i^+ \leq u_i \leq \Phi_i$, and $\Psi_i^+ \equiv \Phi_i$ on $\partial\Omega$, (i) easily follows. Moreover, using the same inequality and the very definition of directional derivative, we obtain, in distributional sense,

$$\frac{\partial \Psi_i^+}{\partial \nu} \geq \frac{\partial u_i}{\partial \nu} \geq \frac{\partial \Phi_i}{\partial \nu},$$

and also (ii) follows. ■

Now we are ready to prove Theorem 8.2.

Proof of Theorem 8.2: let U satisfy the assumptions of the theorem. By the first part of the previous remark $u_i \in L^\infty(\Omega)$ for every i , and hence there exists a constant M such that

$$-\Delta u_i \leq f(x, u_i) \leq M.$$

Moreover, by Proposition 6.3, there exist k -uples (Ψ_i) , (Φ_i) such that $\Psi_i^+(x) \leq u_i(x) \leq \Phi_i(x)$ on $\overline{\Omega}$, $\Psi_i^+(x) = u_i(x) = \Phi_i(x)$ on $\partial\Omega$ and, as we just observed, $\Psi_i, \Phi_i \in W^{1,\infty}(\overline{\Omega})$ for every i .

As usually we define $\phi(x, r)$ as in (19) and we assume by contradiction the existence of a sequence (x_n, r_n) such that $\phi(x_n, r_n) \rightarrow +\infty$. Assume that (up to a subsequence) $\frac{\text{dist}(x_n, \partial\Omega)}{r_n} \geq 1$. This means that $B(x_n, r_n) \cap \partial\Omega = \emptyset$ for every n . In this situation, one can repeat exactly the same proof of Theorem 8.1 (roughly speaking, in such a situation

the blow-up procedure does not “see” the boundary), obtaining the same contradictions. Hence we can assume that $\frac{\text{dist}(x_n, \partial\Omega)}{r_n} \leq 1$ and, by Lemma 8.2,(ii), without loss of generality we can choose $x_n \in \partial\Omega$; moreover, we take r_n such that the inequality in Lemma 8.2,(i) holds. Let $\phi(x_n, r_n) =: L_n^2 \rightarrow +\infty$ and define, for every i ,

$$\begin{aligned} u_{i,n}(x) &:= \frac{1}{r_n L_n} (u_i(x_n + r_n x) - u_i(x_n)), \\ \Psi_{i,n}(x) &:= \frac{1}{r_n L_n} (\Psi_i^+(x_n + r_n x) - \Psi_i^+(x_n)), \\ \Phi_{i,n}(x) &:= \frac{1}{r_n L_n} (\Phi_i(x_n + r_n x) - \Phi_i(x_n)). \end{aligned}$$

We have that $\Psi_{i,n}(x) \leq u_{i,n}(x) \leq \Phi_{i,n}(x)$ on $\bar{\Omega}_n$, where $\Omega_n := \{x \in B(0, 1) : x_n + r_n x \in \Omega\}$ (this inequality holds because the non-scaled functions coincide in $x_n \in \partial\Omega$). We observe that, taking into account that $r_n \rightarrow 0$ and $L_n \rightarrow \infty$, we have

$$-\Delta u_{i,n}(x) = -\frac{r_n}{L_n} \Delta u_i(x_n + r_n x) \leq \frac{r_n}{L_n} M \leq 1$$

for every i , when n is sufficiently large. Testing the above inequality on $u_{i,n}$ we obtain

$$\int_{\Omega_n} |\nabla u_{i,n}|^2 \leq \int_{\Omega_n} u_{i,n} + \int_{\partial\Omega_n} u_{i,n} \cdot \frac{\partial u_{i,n}}{\partial \nu} \leq \|u_{i,n}\|_{L^\infty(\Omega_n)} + \|u_{i,n}\|_{L^2(\partial\Omega_n)} \left\| \frac{\partial u_{i,n}}{\partial \nu} \right\|_{L^2(\partial\Omega_n)}.$$

Our aim is to prove that the righthand side of the previous inequality tends to 0 for every i . This will provide a contradiction with the fact that, by construction, $\sum \int_{\Omega_n} |\nabla u_{i,n}|^2 = 1$ for every n .

Clearly $\Psi_{i,n}(0) = \Phi_{i,n}(0) = 0$. Moreover,

$$\|\nabla \Phi_{i,n}\|_{L^\infty(\Omega_n)} = \frac{1}{L_n} \|\nabla \Phi_i\|_{L^\infty(\Omega \cap B(x_n, r_n))} \rightarrow 0,$$

and the same holds for $\Psi_{i,n}$. This implies $\|\Phi_{i,n}\|_{W^{1,\infty}(\bar{\Omega}_n)} \rightarrow 0$, $\|\Psi_{i,n}\|_{W^{1,\infty}(\bar{\Omega}_n)} \rightarrow 0$ and therefore

$$\|u_{i,n}\|_{L^\infty(\Omega_n)} \rightarrow 0 \quad \forall i.$$

Since $\partial\Omega$ is of class C^1 , we obtain that $\partial\Omega_n$ has bounded $(N-1)$ -dimensional measure, and thus also

$$\|u_{i,n}\|_{L^2(\partial\Omega_n)} \rightarrow 0 \quad \forall i.$$

Therefore the only thing that remains to prove is that $\left\| \frac{\partial u_{i,n}}{\partial \nu} \right\|_{L^2(\partial\Omega_n)}$ is bounded. To this aim, let $\partial\Omega_n = \Gamma_{1,n} \cup \Gamma_{2,n}$, where $\Gamma_{1,n} := \partial B(0, 1) \cap \bar{\Omega}_n$ and $\Gamma_{2,n} := \partial\Omega_n \setminus \Gamma_{1,n}$. Then the estimate on $\Gamma_{1,n}$ descends from Lemma 8.2,(i), recalling that it implies

$$\int_{\Gamma_{1,n}} \left| \frac{\partial u_{i,n}}{\partial \nu} \right|^2 \leq \frac{1}{r_n^{N-1} L_n^2} \int_{\partial B(x_n, r_n) \cap \Omega} |\nabla u_i|^2 \leq \frac{N}{r_n^N L_n^2} \int_{B(x_n, r_n) \cap \Omega} |\nabla u_i|^2 = N.$$

On the other hand, the estimate on $\Gamma_{2,n}$ is an easy consequence of the bounded measure of $\Gamma_{2,n}$ and of Remark 8.1,(ii). ■

9 Further regularity in dimension $N=2$

9.1 Vanishing of the gradient at multiple points

Let $N = 2$ and $U \in \mathcal{S}$. The main goal of this section is to prove that the gradient of U vanishes continuously at points of multiplicity at least three. This result will be established through the application of a monotonicity formula with three or more phases. To start with, we need the following technical result, that allows us to reduce the u_i 's to solutions to suitable divergence-type equations.

Lemma 9.1 *Let $a \in L^\infty$ and let v be an H^1 solution of $-\Delta v \leq a(x)v$ in $B(x_0, r)$. Then, if r is small enough, there exists $\varphi \in C^1$ such that φ is radial with respect to x_0 , $\inf_{B(x_0, r)} \varphi > 0$ and*

$$-\operatorname{div}(\varphi^2(\nabla \frac{v}{\varphi})) \leq 0 \quad \text{in } B(x_0, r).$$

Proof: let us consider the eigenvalue problem

$$\begin{cases} -\Delta u(x) = a(x)u(x) & x \in B(x_0, r) \\ u(x) > 0 & x \in B(x_0, r) \end{cases}$$

If r is small enough, the above problem can be solved in the class of radial functions with respect to x_0 . Let φ be such a solution: now by elementary computations

$$-\varphi^2 \Delta \frac{v}{\varphi} - 2\varphi \nabla \frac{v}{\varphi} \cdot \nabla \varphi \leq 0$$

giving the required inequality. ■

This local reduction will be widely exploited throughout the present and the next section. As a first application it allows to prove a variant of the original monotonicity formula by Alt–Caffarelli–Friedman [2].

Lemma 9.2 *Let $U \in \mathcal{S}$ and $w_i = \sum_{j \in I_i} u_j$, where $I_1 \cup \dots \cup I_h \subset \{1, \dots, k\}$. Assume that $x_0 \in \partial(\{w_i > 0\})$. Then for all $h \geq 2$ there exists C , independent of x_0 such that*

$$\prod_{i=1}^h \frac{1}{r^h} \int_{B(x_0, r)} |\nabla w_i(x)|^2 dx \leq C. \quad (24)$$

Proof: (sketch). Let $x_0 \in \partial\{u_i > 0\}$ and recall that, since $U \in \mathcal{S}$, it holds $-\Delta u_i \leq f_i(u_i)$ for all i . Let $a(x) := \max_{i=1, \dots, k} \left| \frac{f_i(u_i(x))}{u_i(x)} \right|$ and note that $a \in L^\infty$ by the assumption on f_i . By Lemma 9.1, there exists $r > 0$ and a regular radial function φ which is strictly positive on $B(x_0, r)$ such that, for all i

$$-\operatorname{div}(\varphi^2(\nabla \frac{u_i}{\varphi})) \leq 0 \quad \text{in } B(x_0, r).$$

Now consider $w_i = \sum_{j \in I_i} u_j$, where $I_1 \cup \dots \cup I_h \subset \{1, \dots, k\}$ and let $\tilde{w}_i := w_i/\varphi$: then $-\operatorname{div}(\varphi^2(\nabla \tilde{w}_i)) \leq 0$. Let $N = 2$ and set

$$\Phi(r) = \prod_{i=1}^h \frac{1}{r^h} \int_{B(x_0, r)} \varphi^2(x) |\nabla \tilde{w}_i(x)|^2 dx.$$

Following the proof of Lemma 5.1 in [2] we can compute

$$\Phi'(r) \geq h \frac{\Phi(r)}{r} \left(-h + \frac{2}{h} \inf_{\mathcal{P}} \sum_{i=1}^h \sqrt{\Lambda_i} \right)$$

where $\mathcal{P} = \{(\Gamma_i)_{i=1}^h, \cup \Gamma_i = \partial B(0, 1), \operatorname{meas}(\Gamma_i \cap \Gamma_j) = 0\}$ and

$$\Lambda_i = \inf_{v \in H_0^1(\Gamma_i)} \frac{\int_{\Gamma_i} |\partial_\theta v|^2}{\int_{\Gamma_i} |v|^2}.$$

Since $\Lambda_i \geq \left(\frac{\pi}{|\Gamma_i|}\right)^2$ we immediately obtain that $\Phi'(r) \geq 0$ in $[0, r']$.

By this formula and since there exist positive $a < b$ such that $a < \varphi(x) < b$ for all $x \in B(x_0, r')$, it follows in particular that (24) holds. \blacksquare

Now we are ready to prove the main result of this section.

Theorem 9.1 *If $x_0 \in \mathcal{Z}_3$ then $|\nabla U(x)| \rightarrow 0$ as $x \rightarrow x_0$.*

Proof: assume by contradiction the existence of $x_n \rightarrow x_0$ and $r_n \rightarrow 0$ such that

$$\lim_{n \rightarrow +\infty} \frac{1}{r_n^2} \int_{B(x_n, r_n)} |\nabla U|^2 = \alpha \quad (25)$$

$\alpha \in (0, \infty]$.

We first claim that equation (25) holds for $x_n \in \mathcal{Z}_3$. The proof of this fact can be done as follows: let $\rho, r > 0$ be fixed and let $A_{\rho, r} = \{x \in B(\rho, x_0) : d(x, \mathcal{Z}_3(U)) \geq r\}$. By Proposition 6.1, in $A_{\rho, r}$ we can give alternate positive and negative sign to the u_i 's in such a way that the resulting function v locally solves an equation of the form $-\Delta v = f(x, v)$ where $f(x, v(x)) = f_i(x, v(x))$ if $v(x) = \pm u_i(x)$. For $x \in A_{\rho, r}$ we define

$$\Phi(x) = \frac{1}{r^2} \int_{B(x, r)} |\nabla U(y)|^2 dy = \frac{1}{r^2} \int_{B(0, r)} |\nabla U(x + y)|^2 dy.$$

By elementary computations $-\Delta(|\nabla v(x)|^2) = -2|\nabla v(x)|^2 - 2f_x(x, v)|\nabla v(x)| - 2f_s(x, v)|\nabla v(x)|^2$: recalling that v is bounded by Remark 6.2 and since $|\nabla U| = |\nabla v|$, this gives $-\Delta \Phi \leq a\Phi$ on $A_{\rho, r}$, for some positive constant a (depending on r). Then, by an extension of the maximum principle, there exists $C > 0$ (independent of r) such that $\max_{A_{\rho, r}} \Phi \leq C \max_{\partial A_r} \Phi$. This implies that for n large enough, (25) holds with $\alpha = \alpha/2$ and for a choice of x'_n such that $d(x'_n, \mathcal{Z}_3(U)) \asymp r_n$; then taking $z_n \in \mathcal{Z}_3(U)$ such that $|z_n - x'_n| = d(x'_n, \mathcal{Z}_3(U))$ we

obtain that (25) holds for balls centered at z_n and radius $r_n + d(x'_n, \mathcal{Z}_3(U)) \asymp r_n$. Furthermore, by exploiting the Lipschitz regularity of U , we have

$$\int_{\partial B(x_n, \rho_n)} |\nabla U|^2 \leq \frac{3}{\rho_n} \int_{B(x_n, \rho_n)} |\nabla U|^2 \quad (26)$$

where $\rho_n = (2L/C)r_n$ and L is the Lipschitz constant in $\Omega' = \{x \in \Omega : d(x, \partial\Omega) \geq d(x_0, \partial\Omega)/2\} \supset \{x_n\}$.

Hence, in the following let us assume that (25) holds for a choice of $x_n \in \mathcal{Z}_3$ and of radii ρ_n satisfying (26) (we denote ρ_n again by r_n).

Let $r > 0$: by the monotonicity formula with three phases, there exists $C > 0$ (independent of r) such that

$$\prod_{i=1}^3 \frac{1}{r^3} \int_{B(x_n, r)} |\nabla \omega_i(x)|^2 dx \leq C$$

for all $\omega_1 := u_i$, $\omega_2 := u_j$ and $\omega_3 := \sum_{h \notin \{i, j\}} u_h$, such that $x_n \in \partial\{u_i > 0\} \cap \partial\{u_j > 0\} \cap \partial\{u_l > 0\}$ for some $l \notin \{i, j\}$. Then, due to assumption (25), we deduce that there exist at most two components, say u_1 and u_2 , such that

$$\lim_{n \rightarrow \infty} \frac{1}{r_n^2} \int_{B(x_n, r_n)} |\nabla u_i(x)|^2 dx > 0, \quad i = 1, 2 \quad \frac{1}{r_n^2} \int_{B(x_n, r_n)} |\nabla u_i(x)|^2 dx \rightarrow 0 \quad \forall i \geq 3. \quad (27)$$

Let us set

$$L_n^2 := \frac{1}{r_n^2} \int_{B(x_n, r_n)} |\nabla U(x)|^2 dx$$

and consider the sequence of functions

$$U_n(x) = \frac{1}{L_n r_n} U(x_n + r_n x) \quad (28)$$

defined in $x \in B(0, 1)$. Note that, since $U \in \mathcal{S}_{M, k}^*$ by (16), then $U_n \in \mathcal{S}_{M_n, k}^*$ where $M_n := \frac{r_n}{L_n} M$. Then $\int_{B(0, 1)} |\nabla U_n|^2 = 1$ and, by (26), $\int_{\partial B(0, 1)} |\nabla U_n|^2$ is bounded too. By Proposition 7.1 Part (c), there exists $\overline{U} \in H^1(B(0, 1))$ such that, up to subsequences, $U_n \rightharpoonup \overline{U}$ and $\overline{U} \neq 0$. Furthermore, by Part (a), we know that at least two components of U_n does not vanish to the limit: comparing with (27) we have

$$\overline{U} = (\overline{u}_1, \overline{u}_2, 0, \dots, 0)$$

with $\overline{u}_i \neq 0$ for $i = 1, 2$. Moreover, since $M_n \rightarrow 0$, it holds $\overline{U} \in \mathcal{S}_{0, k}^*$. This in particular implies that $\overline{u}_1 - \overline{u}_2$ is harmonic.

Now, if $\widehat{u}_1(0) > 0$, (resp. $\widehat{u}_2(0) > 0$) then $\overline{u}_1 > 0$ (resp. $\overline{u}_2 > 0$) in $B(0, \bar{r})$ for some $\bar{r} > 0$. This implies $\int_{B(0, \bar{r})} \widehat{u}_{1, n}(x) \geq \alpha$ for some $\alpha > 0$, since it converges to the L^1 -norm of \overline{u}_1 in $B(0, \bar{r})$. We can thus apply Remark 7.1 to the sequence $(\widehat{u}_{1, n})$ obtaining $\widehat{u}_{1, n}(0) \geq \alpha/2$. But this is in contradiction with the fact that, by definition, $\widehat{u}_{1, n}(0) = 0$.

Now set $v = \widehat{u}_1$ and assume that $v(0) = 0$. Standard results on harmonic function (see [21]) imply that $v(r, \theta) \sim r^p \cos p(\theta + \theta_0)$ for some $p \geq 1$. Thus, by the strong convergence in H^1 and a diagonal process, we can assume that, for n large enough and $i = 1, 2$, there exists $m_i > 0$ such that $u_{n, i} > m_i$ on a circular sector $A_{n, i} = \{(\rho, \theta) : r < \rho < R, \alpha_{n, i} < \theta <$

$\beta_{n,i}\} \subset \{u_{n,i} > 0\}$. Here we can assume, for instance, $\alpha_{n,1} = \alpha < \beta_{n,1} < \alpha_{n,2} < \beta_{n,2} = \beta$ with α and β fixed; note that $\alpha_{n,2} - \beta_{n,1} \rightarrow 0$ as $n \rightarrow \infty$. Now, since 0 is a zero of U_n with multiplicity $m(0) \geq 3$, there exists a third component, say $u_{n,3}$, and a continuous path $\gamma_n : [0, 1] \rightarrow B(0, R)$ such that $\gamma_n(0) = 0$, $\gamma_n(1) \in \partial B(0, R)$, $\gamma_n(t) \in \{\beta_{n,1} < \theta < \alpha_{n,2}\}$, $u_{n,3}(\gamma_n(t)) > 0$ for all $t \in (0, 1]$. Therefore, if we set $S = \{(\rho, \theta) : r < \rho < R, \alpha < \theta < \beta\}$ and denote by $\omega_{n,i}$ ($i = 1, 2$) the connected component of $\{u_{n,i} > 0\}$ that contains $A_{n,i}$, then $\text{dist}(S \cap \omega_{n,1}, S \cap \omega_{n,2}) > 0$.

Now consider

$$T_n(x) = r_n x + x_n$$

and for $x \in T_n(S)$ define

$$w_n(x) = \begin{cases} u_i(x), & x \in T_n(\omega_{n,i}) \quad i = 1, 2 \\ -u_i(x), & x \in \{u_i > 0\} \setminus (T_n(\omega_{n,1}) \cup T_n(\omega_{n,2})) \quad i = 1, \dots, k \end{cases}$$

We claim that

$$-\Delta w_n \geq f(w_n) \quad \text{in } T_n(S) \quad (29)$$

where $f(x, s) := f_i(x, s)$ for $x \in \{u_i > 0\}$.

In order to prove this assertion, let us drop the dependence on n . Then fix any $\phi > 0$, $\phi \in C_0^1(T(S))$: we have to prove

$$\int_{T(S)} (\nabla w \nabla \phi - f(w)\phi) dx > 0.$$

For easier notation, set $A_1 = T(\omega_1)$, $A_2 = T(\omega_2)$ and $B = T(S) \setminus (A_1 \cup A_2)$. Now take a partition of the unity in such a way that $\phi(x) = \phi_1(x) + \phi_2(x)$ where $\phi_i > 0$ and $\{\phi_i > 0\} \subset A_i \cup B$. For $x \in T(S)$ define the function

$$w_1(x) = \begin{cases} u_1(x), & x \in A_1 \cup (B \cap \{u_1 > 0\}) \\ -u_i(x), & x \in \{u_i > 0\} \quad i = 2, \dots, k \end{cases}$$

(Analogous definition for w_2 .) Since $U \in \mathcal{S}$ (namely $-\Delta \hat{u}_1 \geq f(\hat{u}_1)$), it holds

$$\int_{T(S)} (\nabla w_1 \nabla \phi_1 - f(w_1)\phi_1) dx > 0$$

and

$$\int_{T(S)} (\nabla w_2 \nabla \phi_2 - f(w_2)\phi_2) dx > 0.$$

Summing up the two inequalities we obtain

$$\int_{T(S)} (\nabla w \nabla \phi - f(w)\phi) dx > -2 \sum_{i=1,2} \int_B (\nabla u_i \nabla \phi_i - f(u_i)\phi_i) dx.$$

As we are going to prove, each term of the above sum is negative, and this finally proves assertion (29). To this aim let $i = 1$, and introduce a cut-off function η such that $\eta = 1$ on $B \cap \{u_1 > 0\}$ and $\eta = 0$ on $T(S) \setminus B$. Let $\psi = \eta\phi_1 \in C_0^1(T(S))$; then

$$\int_B (\nabla u_1 \nabla \phi_1 - f(u_1)\phi_1) dx = \int_{T(S)} (\nabla u_1 \nabla \psi - f(u_1)\psi) dx.$$

Then the righthand side is negative since $U \in \mathcal{S}$ (namely $-\Delta u_1 \leq f(u_1)$).

Final step. Now fix $t_n > 0$ and $R > r_n > r$ in such a way that, if we set $y_n = \gamma_n(t_n)$ then it holds $\int_{\partial B(y_n, r_n)} w_n \geq \alpha$ for some positive α . (This is due to the fact that $\partial B(y_n, r_n) \subset \omega_{n,1} \cup \omega_{n,2}$ except for a small piece of total length less than $2R(\alpha_{n,2} - \beta_{n,1}) \rightarrow 0$ as $n \rightarrow \infty$.) This allows to apply Remark 7.1 to the sequence of rescaled functions $(\frac{w_n}{r_n L_n})$ (as in (28)), since they satisfy $\Delta \frac{w_n}{r_n L_n} \geq -M_n \rightarrow 0$. This gives $w_n(y_n) > 0$, in contradiction with the fact that, by construction, $w_n(y_n) = -u_{n,3}(y_n) < 0$. ■

9.2 Local properties of the free boundary

Let again $N = 2$; the purpose of this section is to deepen the investigation of the behavior of the elements U of the class \mathcal{S} around multiple points, that is, $x \in \Omega$ with $m(x) \geq 2$. Our main goal is to prove that, near a multiple point, U and its null set exhibit the same qualitative behavior of harmonic functions and their nodal sets. We refer the readers to the fundamental papers of Alessandrini [1] and Hartman Winter [21] for the main results about the zero set of harmonic functions and, more in general, of functions in the kernel of a divergence type operator. We shall obtain the desired description for those $U \in \mathcal{S}$ with the property that each component has connected support, i.e.

$$U \in \mathcal{S} \text{ such that } \{u_i > 0\} \text{ is connected } \forall i. \quad (30)$$

In order to simplify the topological arguments involved in the proof of our results, in what follows we shall make the additional (but not necessary) assumptions that Ω is simply connected and that all the boundary data are not identically zero, namely $\phi_i \not\equiv 0$ on $\partial\Omega$. In this situation the assumption (30) (and Remark 6.1) implies that $\{u_i > 0\}$ is simply connected.

Let us start our description by analyzing points of multiplicity two, that will be denoted by

$$\mathcal{Z}^2 = \{x \in \Omega : m(x) = 2\}.$$

We already noticed in Remark 6.1 that this set is locally a regular \mathcal{C}^1 arc around those points where ∇U does not vanish. Our first result states that in fact this is always true:

Lemma 9.3 *Let $x_0 \in \Omega$ such that $m(x_0) = 2$. Then $\nabla U(x_0) \neq 0$ and \mathcal{Z}^2 is locally a \mathcal{C}^1 -curve through x_0 .*

Proof: since $x_0 \in \partial\{u_1 > 0\} \cap \partial\{u_2 > 0\}$, then for all r small enough $B(x_0, r) \cap \{u_i > 0\} = \emptyset$ for all $i > 2$. Assume by contradiction $\nabla u_1(x_0) = 0 = \nabla u_2(x_0)$; then $u = u_1 - u_2$

is C^1 and satisfies the equation $-\Delta u = a(x)u$ (with $a(x) = \frac{f_1(u_1(x)) - f_2(u_2(x))}{u_1(x) - u_2(x)}$) in $B(x_0, r)$. By Lemma 9.1 there exists a positive, regular function φ which is radial with respect to x_0 such that

$$-\operatorname{div} \left(\varphi^2 \nabla \left(\frac{u}{\varphi} \right) \right) = 0$$

on $B(x_0, r)$. Then u/φ satisfies all the assumptions necessary to apply the main theorem in [1], which says that the null level set of u/φ (indeed of u) near x_0 is made up by a finite number of curves starting from x_0 . Obviously in our situation such number must be even. Now recall that each $\{u_i > 0\}$ is connected in Ω : by a geometrical argument we can see that the null level set is made up by (two semi-curves joining in) one C^1 -curve. But again applying [1] we have $\nabla U(x_0) \neq 0$, a contradiction. ■

Lemma 9.4 *Let $x_0 \in \mathcal{Z}_3$. Then there exists $\{x_n\} \subset \Omega$ such that $m(x_n) = 2$ and $x_n \rightarrow x_0$*

Proof: assume not, then there would be an element y_0 of \mathcal{Z}_3 having a positive distance d from \mathcal{Z}^2 . Let $r < d/2$: then the ball $B(y_0, r)$ intersects at least three supports; therefore there exist, say, $x \in \{u_i > 0\}$ and $z_0 \in \mathcal{Z}_3$ such that $\rho = d(x, z_0) = d(x, \mathcal{Z}_3) < d(x, \mathcal{Z}^2)$. Then the ball $B(x, \rho)$ is tangent from the interior of $\{u_i > 0\}$ to \mathcal{Z}_3 in z_0 ; furthermore u_i solves an elliptic PDE and it is positive on its support: we thus infer from the Boundary Point Lemma that $\nabla u_i(z_0) \neq 0$, in contrast with Theorem 9.1. ■

Let us now prove an asymptotic formula describing the behavior of $\sum u_i$ in the neighborhood of a multiple point which is isolated in \mathcal{Z}_3 .

Theorem 9.2 *Let $x_0 \in \Omega$ with $m(x_0) = h \geq 3$ that is isolated in \mathcal{Z}_3 . Then there exists $\theta_0 \in (-\pi, \pi]$ such that*

$$U(r, \theta) = r^{\frac{h}{2}} \left| \cos\left(\frac{h}{2}(\theta + \theta_0)\right) \right| + o(r^{\frac{h}{2}})$$

as $r \rightarrow 0$, where (r, θ) denotes a system of polar coordinates around x_0 .

Proof: by assumption x_0 is isolated in \mathcal{Z}_3 , then there is $B = B(x_0, \rho)$ such that $\mathcal{Z}_3 \cap B = \{x_0\}$; furthermore since each $\{u_i > 0\}$ is simply connected, then $\{u_i > 0\} \cap \partial B = b_i$ is a connected arc on ∂B for the h indices involved in x_0 . Choosing a slightly smaller radius we can suppose that the intersection of ∂B with \mathcal{Z}^2 is transversal (recall that, by Lemma 9.3, \mathcal{Z}^2 is locally a C^1 -curve). Let us assume that h is even: then we define a function $v(r, \theta)$ such that $|v(r, \theta)| = |u(r, \theta)|$ and $\operatorname{sign}(v(r, \theta)) = (-1)^j$ if $(\rho, \theta) \in b_j$, $j = 1, \dots, h$. Note that the resulting function is alternately positive and negative on the consecutive (with respect to θ) local components of U . If on the contrary h is odd, we define $|v(r, \theta)| = |u(r^2, 2\theta)|$ and we prescribe an alternating sign to the local components of $u(\rho^2, 2\theta)$. It is worthwhile noticing that the resulting function v is of class C^1 in $\tilde{B} = B(x_0, \rho^2)$: indeed, $\{u_i > 0\} \cap B$ is simply connected for every i and thus each connected component of $B \setminus u^{-1}(0)$ corresponds to two components of $\tilde{B} \setminus v^{-1}(0)$ to which we give opposite sign. In both the even and the odd cases v is of class C^1 and it solves an equation of type $-\Delta v = a(x)v$ in $B \setminus \{x_0\}$ (resp. $\tilde{B} \setminus \{x_0\}$), where $a \in L^\infty$ is given by $f(v)/v$ and $r^2 f(v)/v$ respectively. Moreover

$\nabla v(x_0) = 0$ by Theorem 9.1: this implies that v is in fact solution of the equation on the whole of B (resp. \tilde{B}) and thus it is of class $C^{2,\alpha}$. Now, choosing r small enough, by Lemma 9.1 we have a positive, regular function φ such that

$$-\operatorname{div} \left(\varphi^2 \nabla \left(\frac{v}{\varphi} \right) \right) = 0.$$

Then, we can apply to v/φ the asymptotic formula of Hartman and Winter as recalled in [21]. To complete the proof, let us observe that h represents the number of connected components of $\cup\{u_i > 0\}$ in a ball centered in x_0 , providing the choice of the periodicity of the cosine in the representation formula. ■

The last part of this section is devoted to prove that \mathcal{Z}_3 consists of (a finite number of) isolated points: thus Theorem 9.2 will provide the complete description of U near multiple points.

We start by proving an intermediate result:

Proposition 9.1 *\mathcal{Z}_3 has a finite number of connected components.*

Proof: let $\omega_i := \{u_i > 0\}$; take an index pair (i, j) such that $\partial\omega_i, \partial\omega_j$ do intersect and we consider

$$\begin{aligned} \Gamma_{i,j} &= \partial\omega_i \cap \partial\omega_j \cap \mathcal{Z}^2 \\ \omega_{i,j} &= \omega_i \cup \omega_j \cup \Gamma_{i,j}. \end{aligned}$$

Since by Lemma 9.3 $\Gamma_{i,j}$ is locally a regular arc and each ω_i is open, it easily follows that $\omega_{i,j}$ is open. Furthermore $\omega_{i,j}$ is simply connected. Indeed, let us consider a loop γ in $\omega_{i,j}$ which is not contractible in $\omega_{i,j}$. This means that Ω_γ , the internal region of the loop, contains at least a multiple intersection point. Since each support is connected, there exists ω_h , $h \neq i, j$ such that $\omega_h \subset \Omega_\gamma$. But this is in contradiction with the fact that $\omega_h \cap \partial\Omega = \{\phi_h > 0\}$.

A similar reasoning allows to prove that each $\Gamma_{i,j}$ consists in a single C^1 -arc: as straightforward consequence of these facts the set of multiple points can have only a finite number of connected components. ■

Now we will need the following definition of adjacent supports:

Definition 9.1 *We say that ω_i and ω_j are adjacent if*

$$\Gamma_{i,j} \neq \emptyset.$$

Let us list some basic properties:

1. Every ω_i is adjacent to some other ω_j . This follows from the Boundary Point Lemma.
2. Let us pick k points $x_i \in \omega_i$, $i = 1, \dots, k$. If ω_i and ω_j are adjacent, $i < j$, then there exists a smooth arc γ_{ij} with $\gamma_{ij}(0) = x_i$, $\gamma_{ij}(1) = x_j$ lying in $\omega_i \cup \omega_j \cup \{y_{ij}\}$, for some $y_{ij} \in \mathcal{Z}^2$.

3. We can choose the arcs γ_{ij} in a manner that they are mutually disjoint, except for the extreme points.

We call \mathcal{G} the graph induced by the arcs γ_{ij} and their endpoints.

Construction of an auxiliary function v . Aim of this paragraph is to build up by the components of U a C^1 function carrying a sign law which is compatible with the adjacency relation and solves an elliptic equation. Let us first assume that

\mathcal{G} has no loops.

In this situation the graph is a disjoint union of a finite number of branches: we select one of them. Now we define v as follows: if the index i is not involved in the branch, we set $v \equiv 0$ in $\{u_i > 0\}$. Next we follow the selected branch of the graph and prescribe a sign to each vertex by alternating plus and minus. We then define $v(x) = \pm u(x)$ according to this sign rule. Taking into account Remark 6.1 and Theorem 9.1, with this procedure we obtain a C^1 -function on Ω .

Moreover v solves an elliptic equation as follows: let us define $f(x, s) := f_i(x, s)$ if $x \in \{u_i > 0\}$, $i = 1, \dots, k$. Then by construction

$$-\Delta v = a(x)v \quad \text{in } \Omega \setminus \mathcal{Z}_3 \quad (31)$$

where $a = f(v)/v \in L^\infty$. In fact we are going to prove that v solves the elliptic equation on the whole of Ω . We need a technical lemma

Lemma 9.5 *Let $\mathcal{Z}_3^\varepsilon = \{x \in \Omega : m(x) = 2\} \cap B_\varepsilon(\mathcal{Z}_3)$. For every $\delta > 0$ there exists $\varepsilon > 0$ such that*

$$\int_{\mathcal{Z}_3^\varepsilon} |\nabla v| < \delta.$$

Proof: by testing the equation (31) with the test function $\varphi = 1$ and integrating over the set $u_i > \alpha$ we obtain the bound, independent of α and i ,

$$\int_{\partial\{u_i > \alpha\}} |\nabla v| < C$$

and therefore, passing to the limit as $\alpha \rightarrow 0$,

$$\int_{\partial\{u_i > 0\}} |\nabla v| < C.$$

The assertion then follows from Lemma 9.4, together with Proposition 9.1. ■

Lemma 9.6 *v solves $-\Delta v = a(x)v$ in Ω .*

Proof: let us fix a connected component of \mathcal{Z}_3 , named A . Thanks to Lemma 9.4, for any ε we can take a neighborhood of A , say $\mathcal{V}_\varepsilon \subset B_\varepsilon(\mathcal{Z}_3)$, in such a way that the boundary $\partial\mathcal{V}_\varepsilon$ is the union of a finite number of arcs of \mathcal{Z}^2 and supplementary union of pieces of total length smaller than $C\varepsilon$. Let φ be a test function. We write

$$\begin{aligned} \int_\Omega (\nabla v \nabla \varphi - a(x)v\varphi) &= \\ &= \int_{\Omega \setminus \mathcal{V}_\varepsilon} (\nabla v \nabla \varphi - a(x)v\varphi) + \int_{\mathcal{V}_\varepsilon} (\nabla v \nabla \varphi - a(x)v\varphi) \leq \\ &\leq C \int_{\partial\mathcal{V}_\varepsilon} |\nabla v| + C \int_{\mathcal{V}_\varepsilon} (|\nabla v| + |v|). \end{aligned}$$

Let $\delta > 0$: we can find $\varepsilon > 0$ such that Lemma 9.5 holds. Moreover, by Theorem 9.1, we can assume that ε is taken so small that $\sup_{\mathcal{V}_\varepsilon} (|\nabla v| + |v|) < \delta$. Hence the above integral is bounded by $C\delta$. Since δ was arbitrarily chosen we obtain that v solves the equation in a distributional sense. Usual regularity arguments allow us to complete the proof. ■

With this we easily deduce the final result, that is, U has only a finite number of multiple points:

Lemma 9.7 *The set \mathcal{Z}_3 consists of a finite number of points.*

Proof: since by Lemma 9.6 v is a solution of $-\Delta v = a(x)v$ in Ω , locally we can reduce v to a function in the kernel of a divergence-type operator as in Lemma 9.1. Then the results of [1] and [21] ensures that v has only a finite number of multiple points. ■

Let us now go back to the case when the graph associated to U presents loops:

\mathcal{G} has a loop.

Let us first define an order relation between loops, according whether one is contained in the interior region of the other. Let us select a *minimal* loop γ (no other loops are contained in Ω_γ , its interior region). We can assume that Ω_γ contains at least an element $x_0 \in \mathcal{Z}_3$ (if not, we can perform a conformal inversion exchanging the inner with the outer points): fix x_0 as the origin. Note that the supports involved in a minimal loop have the remarkable property that, for all i , ω_i is adjacent to ω_j for only two indices different from i . Thanks to this property, if the number of vertex of γ is even, we manage to assign a sign law to all the subset of \mathcal{G} contained in Ω_γ so that adjacent supports have opposite sign: it suffices to follow the loop and prescribe alternating sign to its vertices. We define $v(x) = \pm u(x)$ according to this law.

If the number of vertex of γ is odd, we wish to “double” the loop. To this aim, we define new ω_i ’s by taking the complex square roots of the old ones. In this way the new loop γ will have an even number of edges and we define $v(r, \theta) = \pm u(r^2, 2\theta)$ by giving alternating sign at the vertices of the loop. In this way we define a function v in Ω_γ which is of class C^1 thanks to Remark 6.1 and Theorem 9.1. Again, v solves the elliptic equation

$$-\Delta v = a(x)v \quad \text{in } \Omega_\gamma \setminus \mathcal{Z}_3 \quad (32)$$

where a is either $a = f(v)/v$ or $a = r^2 f(v)/v$ according to the construction of v . As in the previous case, it is possible to show that v solves the equation on the whole of Ω_γ . The

proof follows exactly that of Lemma 9.6, with two remarks; first note that Lemma 9.5 still holds. Furthermore, in this situation the set $\mathcal{Z}_3 \cap \Omega_\gamma$ is connected (assume not; then in Ω_γ there are two points of \mathcal{Z}_3 which are connected by a regular arc of double points. By a simple geometrical argument in the plane this implies that one of the supports is adjacent to other three different supports: as already observed, this is in contradiction with the minimality of the loop). Then \mathcal{V}_ε will be a neighborhood of the whole $\mathcal{Z}_3 \cap \Omega_\gamma$ with the properties required in the proof of that lemma.

This immediately provides

Lemma 9.8 *There is only one point of \mathcal{Z}_3 lying in the interior of a minimal loop γ .*

Proof: let Ω_γ denote the internal region of γ : since by Lemma 9.6 v is a solution of $-\Delta v = a(x)v$ in Ω_γ , locally we can reduce v to a function in the kernel of a divergence-type operator as in Lemma 9.1. Then by [1] and [21] we know that only a finite number of points of \mathcal{Z}_3 lie in Ω_γ . On the other hand, by the minimality of γ , $\mathcal{Z}_3 \cap \Omega_\gamma$ is connected. Thus the origin is the only one multiple point contained in Ω_γ . ■

We can finally prove that multiple points are isolated

Theorem 9.3 *The set \mathcal{Z}_3 consists of isolated points.*

Proof: recalling Proposition 9.1, we argue by induction over the number h of connected components of the set \mathcal{Z}_3 . If $h = 1$ then, by Lemma 9.8 there is at most one minimal loop of the adjacency relation. If there is one, then Lemma 9.8 gives the desired assertion. If there are none, the thesis directly follows by Lemma 9.7. Now, let the Theorem be true for h and assume that \mathcal{Z}_3 has $h + 1$ connected components. Again, if the adjacency relation has no loops we are done. Otherwise, we apply Lemma 9.8 to treat those connected components contained in the interior of the minimal loop and the inductive hypothesis to treat all those contained in the outer region. ■

Remark 9.1 Having proved that the multiple points are isolated, the existence of points of multiplicity zero can be easily ruled out for connected domains.

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